Invertibility of dynamical systems in granular phase space

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The (non)invertibility of iterated dynamical systems is not necessarily preserved under computer discretization. This is shown for linear congruential generators and the circle map. For the latter, the loss of invertibility becomes worse for finer discretization steps.
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Computer simulations (experimental mathematics) have been a crucial tool in the understanding of new phenomena in nonlinear dynamics and complex systems. Classic examples are the Mandelbrot set and the Feigenbaum period-doubling route to chaos. It is not surprising, then, that the effects of computer discretization on dynamics, especially chaotic, have been the subject of much study in recent years for both conservative [1] and dissipative [2,3] systems. An implicit inference of the studies so far has been that finite-state arithmetic preserves the (non)invertibility of the dynamics. Invertibility occurs when only one trajectory leads to each state, this is, each state has exactly one preimage. A good representative discussion for the Chirikov standard map, which is conservative and also invertible, is given in Refs. [1(a)] and [4]: in the discrete representation, regular Kol'mogorov-Arnol'd-Moser surfaces and stochastic regions are well mimicked respectively, by one short and several longer cycles with a distribution of lengths similar to that of a random mapping of integers.

In this paper we show examples where first invertibility and then its opposite are not preserved in granular (discretized) phase space. Systems which are noninvertible in the continuous description can be made invertible upon discretization; this has been turned into an art form in the random number generator (RNG) literature, in which long limit cycles without transients are desirable [5]. After illustrating this, we will summarize results of a study of the circle map,

$$\theta_{n+1} = \theta_n + \Omega - (K/2\pi)\sin(2\pi\theta_n), \tag{1}$$

of interest in physics because of its mode-locking behavior and interesting transitions from quasiperiodicity to chaos [6]. This map is invertible in the continuous case as long as $K<1$ [4], but becomes noninvertible upon discretization of the space $0 \leqslant \theta \leqslant 1$. While in the case of the RNG invertibility only appears with special parameter choices, in the case of the circle map the loss of invertibility becomes worse as the discretization step gets smaller, contrary to what one would expect.

We begin by discussing known features of the discretized representation of a dynamical system $x_{n+1} = f(x_n)$, where x can denote a vector and n is the (integer) time step [7]. This is usually achieved by roundoff or truncation [8] (what a computer would do) of a continuous dynamical system, which essentially turns it into an integer map. Since the variable x can only have a finite number (m) of states, and the function $f(x_n)$ is deterministic, an initial condition can be iterated at most m times before one of the values repeats. This, known as the pigeonhole principle, causes the appearance of limit cycles and possibly transients, the former usually corresponding to unstable cycles of the continuous map which are made stable by discretization, and the latter associated with noninvertible dynamics. Transients are sequences

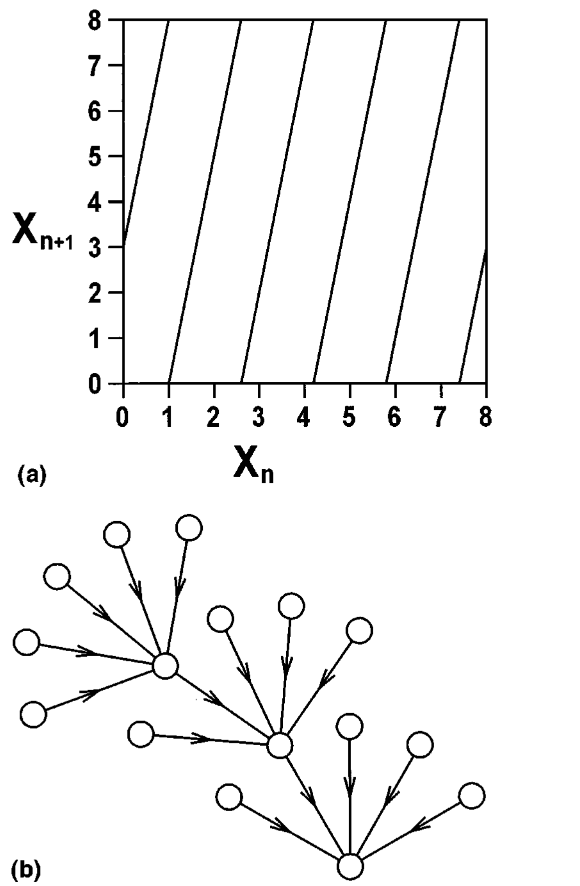

FIG. 1. (a) The continuous map $x_{n+1} = 5x_n + 3$, modulo 8, with (b) its expected de Bruijn diagram.

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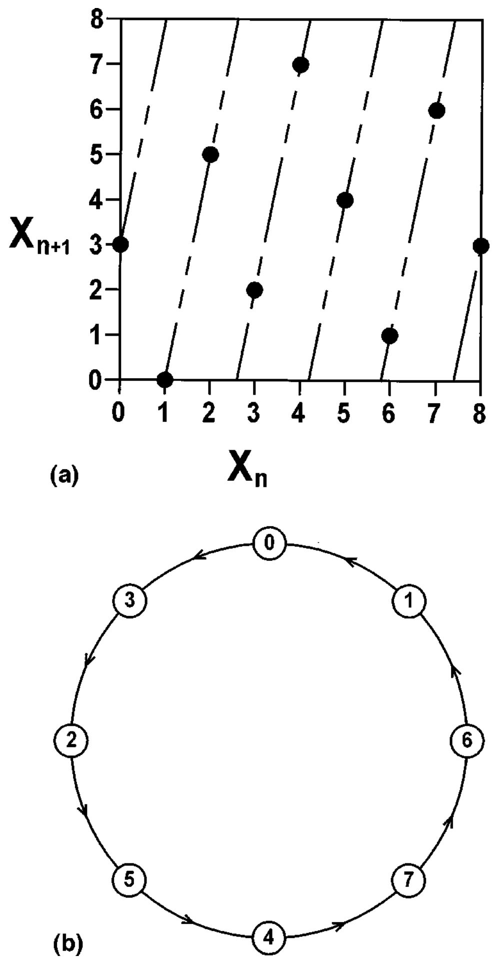

FIG. 2. (a) The linear congruential generator $x_{n+1}=5x_n+3$ modulo 8, with (b) its de Bruijn diagram. The mapping of Fig. 1 has become invertible.

of states which lead to fixed points or limit cycles, but cannot be revisited. Examples are given in Ref. [2], and in Fig. 3 below. The structure of phase space is most usefully visualized in terms of directed graphs or de Bruijn diagrams, such as Figs. 1(b), 2(b), and 3 in this paper.

First, we consider the map $x_{n+1}=ax_n+c$, modulo m, where the meaning of m is as given above. This is the well-known linear congruential RNG. For purposes of illustration we choose the parameter values $a=5$, $c=3$. Figure 1(a) corresponds to the continuous map, where we clearly see that each iterate has five preimages: the map is noninvertible. This should correspond to a de Bruijn diagram as shown in Fig. 1(b): the trees mimic the five-to-one correspondence, and end in fixed points or limit cycles as discussed above. However, under judicious choices of a,c,m [9], the discrete map becomes invertible: one choice is $m=8$. We show the discretization in Fig. 2(a): each value only has one preimage, and moreover, the de Bruijn diagram consists of only one maximal cycle of length m, shown in Fig. 2(b). While the number-theoretical properties of this map are well known, apparently it has not been discussed previously in terms of discrete dynamical systems and de Bruijn diagrams. We note that RNGs have been the subject of recent important studies in the physics literature [10].

The situation is more serious for the circle map; we discuss specifically roundoff, although the conclusions also apply to truncation with minor algebraic changes [8]. We consider this map in the invertible range of K as specified above. The trouble with the circle map occurs near the extreme values $\theta\sim 0$, $\theta\sim 1$, and with $K\sim 1$, which gives slopes $d\theta_{n+1}/d\theta_n\sim 1-K\sim 0$. Under this condition, noninvertible many-to-one mappings are very likely under the effects of discretization. We did a random sampling of fifty combinations in Ω-K parameter space, for $32\leqslant m<2048$, and found only three instances of invertible behavior, all of them for $K\leqslant 0.01$. Only one of them consisted of a single limit cycle, the other two of several small limit cycles. A study of de Bruijn diagrams for $m=32$ confirms the expectation that there are more many-to-one mappings of states for large than for small values of K, resulting in noninvertible behavior and the existence of many limit cycles with transients. An example of noninvertible behavior is given in Fig. 3, where several states lead to state 6. Transients leading to this state from outside the period-5 limit cycle cannot be revisited.

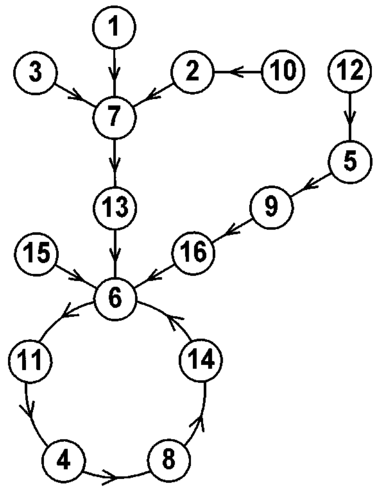

FIG. 3. Example of noninvertible discretized circle map. Label k corresponds to state $\theta=k/16$. Parameter values are $\Omega=0.404\,004$, $K=0.967$.

The conditions for invertibility correspond to limit cycles in which the difference between consecutive iterates is j/m,

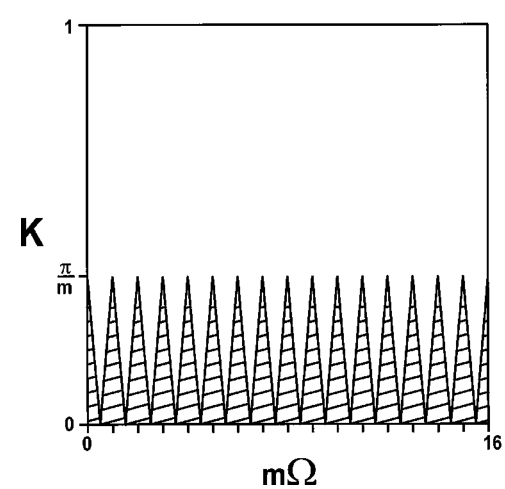

FIG. 4. Regions of invertibility for the circle map in Ω-K parameter space for $m=16$, with roundoff. Note that for $\pi/m\leqslant K\leqslant 1$ the system cannot be invertible. The vertical axis is not shown to scale.

modulo 1, with j integer. If j is a relative prime of m, only one cycle of maximal length will occur, otherwise there will be several cycles. Combining Eq. (1) with the definition of footnote 8, one obtains conditions for Ω and K that guarantee no transients. We obtain $\Omega=(j+\epsilon)/m$, with $|\epsilon|<1/2$, which also requires $K<\pi(1-2|\epsilon|)/m$. Figure 4 shows the regions that satisfy both constraints for invertibility for $m=16$ (shaded areas). Only the triangles with the base centered at odd multiples of $\Omega=1/m$ correspond to single limit cycles, and hence to ergodic behavior. For truncation the diagram is the same, except that the triangles are shifted horizontally by $1/2m$. For other values of m there will also be triangles with bases centered at multiples of $1/m$, and with vertices at $K=\pi/m$; in other words, the area of parameter space in which the system is invertible goes to zero as the discretization step $1/m$ becomes finer. These results have been confirmed by detailed numerical experiments near the boundaries of Fig. 4.

In this paper we have shown a surprising feature of computer discretization of nonlinear maps: invertibility, one of the most fundamental properties of a dynamical system, is not necessarily preserved. This feature is of particular concern for the circle map (and generally for invertible maps with derivatives near zero) in that invertibility actually becomes more difficult to achieve with smaller discretization steps. Finally, the relation between this work and other recent studies of reversibility and symbolic dynamics [11] is under current study.

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[7] Time discretization should not be confused with state-variable discretization.

[8] $\theta_{n+1}\rightarrow k/m$ whenever the iterate of θ_n falls between $(k-1/2)/m$ and $(k+1/2)/m$ (roundoff), or between k/m and $(k+1)/m$ (truncation). These intervals are closed on the right.

[9] The conditions for a maximal cycle are as follows: c is relatively prime to m, $b=a-1$ is a multiple of p for every prime p dividing m, and b is a multiple of 4 if m is a multiple of 4. See, for example, M. Greenberger, J. Assoc. Comput. Mach. **8**, 383 (1961).

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